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Sections of $U(2n)/Sp(s)$ over S^{4n-1}

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ABSTRACT

We describe the sections of $U(2n)/Sp(s)$ over S^{4n-1} in terms of the sections of the symplectic Stiefel manifold $Sp(n)/Sp(s)$ and we express the orders of obstructions to sectioning $U(2n)/Sp(s)$ in terms of orders of obstructions to sectioning $Sp(n)/Sp(s)$. In certain cases we give the exact values of these orders.

1. INTRODUCTION

In [6] we have determined the integers n such that for a given s the fibration $U(n)/Sp(s)\to S^{2n-1}$ admits a cross section. According to these results, when n is even the existence of cross sections depends on the existence of cross sections of the symplectic Stiefel manifold $Sp(m)/Sp(s)$, where $m=n/2$. The purpose of the present paper is to shed some light onto the exact relationship between these two families of cross-sections.

In § 2 the relationship between the sections of $U(2n)/Sp(s)$ and the sections of $Sp(n)/Sp(s)$ is given. This relationship and the description of sections of $U(2n)/Sp(s)$ in terms of the sections of $Sp(n)/Sp(s)$ is expressed as our first main result in Theorem 1. These sections are especially important in the study of almost-quaternion substructures on the sphere and the results of Theorem 1 say something about the construction of these substructures. This is explained in § 3. To make the description of the relationship between sections of

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$U(2n)/Sp(s)$ and $Sp(n)/Sp(s)$ complete, in § 4 we express the orders of obstructions to sectioning $U(2n)/Sp(s)$ in terms of the orders of obstructions to sectioning $Sp(n)/Sp(s)$. This is given as the second main result in Theorem 2. As a consequence of this and by the known results about the orders of obstructions to sectioning $Sp(n)/Sp(s)$, in certain cases we give the exact values of these orders (Theorem 3).

Throughout the paper the following conventions will be used.

Let F denote either $\mathbb{C}$ or $\mathbb{H}$ (quaternions). On the vector space F^n define the inner product $(x/y)=x_1\bar{y}_1+\ldots+x_n\bar{y}_n$ and the norm $\|x\|=(x/x)^{1/2}$ for $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in F^n$. The unitary group $U(n)$ and the symplectic group $Sp(n)$ are the norm preserving automorphisms of $\mathbb{C}^n$ and $\mathbb{H}^n$ respectively. Let e_j be the vector in $\mathbb{R}^n$ which has 1 in j^{th} entry and 0 elsewhere. Similarly we let a_j be the vector in $\mathbb{C}^n$ which has 1 in j^{th} entry and 0 elsewhere. If $s<n$ we shall consider $U(s)$ embedded in $U(n)$ by considering it to be the elements of $U(n)$ which leave $a_{s+1},\ldots,a_n$ fixed. The embedding $Sp(s)\subset Sp(n)$ is similarly defined. Every vector $(z_1,\ldots,z_n)$ in $\mathbb{C}^n$ will be considered as a vector $(y_1,x_1,\ldots,y_n,x_n)$ where $z_r=x_r+iy_r$, and every vector $(q_1,\ldots,q_n)$ in $\mathbb{H}^n$ will be considered as a vector $(w_1,z_1,\ldots,w_n,z_n)$ in $\mathbb{C}^{2n}$ where $q_r=z_r+jw_r$. The embedding $Sp(n)\subset U(2n)$ is defined in accordance with these conventions.

2. THE SECTIONS OF $U(2n)/Sp(s)$

To describe the sections of $U(2n)/Sp(s)$ and the relationship with the sections of $Sp(n)/Sp(s)$ let $1:Sp(n)/Sp(s)\to U(2n)/Sp(s)$ denote the inclusion, $h:U(2n)\to\to U(2n)/Sp(s)$ be the quotient map. Also, let $p:U(2n)\to S^{4n-1}$ be the projection which takes every $2n$-frame $(u_1,\ldots,u_{2n})$ in $\mathbb{C}^{2n}$ to u_{2n}. Denote the fibration $Sp(n)/Sp(s)\to S^{4n-1}$ by q and the fibration $U(2n)/Sp(s)\to S^{4n-1}$ by $\tilde{p}$. Throughout the paper let ι denote the positive generator of the group $\Pi_{4n-1}(S^{4n-1})\cong\mathbb{Z}$. Following [4], we call an element $\alpha\in\Pi_{4n-1}(Sp(n)/Sp(s))$ an m-section if $q(\alpha)=m\iota$, where m is an integer. Similarly we define an m-section of $U(2n)/Sp(s)$ to be an element $\beta\in\Pi_{4n-1}(U(2n)/Sp(s))$ such that $\tilde{p}\#(\beta)=m\iota$. By covering homotopy theorem every 1-section is represented by a cross section. Now, keeping in mind that cross-sections of $U(2n)/Sp(s)$ exist when $Sp(n)/Sp(s)$ has a cross section ([6]), we can state and prove the following theorem.

THEOREM 1. Let η_{s-1} denote the underlying real bundle of the quaternionic Hopf line bundle over $P_{s-1}(\mathbb{H}^n)$, and let c_s denote the order of $J(\eta_{s-1})$ in $J(P_{s-1}(\mathbb{H}^n))$. Assume $c_s|n$. In other words assume $Sp(n)/Sp(s)$ has a cross section. Then

(i) If m is any integer and θ is a $(2n-1)!m+1$ section of $Sp(n)/Sp(s)$, then $1_{\#}(\theta)-mh_{\#}(\alpha)$ is a 1-section of $\tilde{p}$ for a suitable generator α of $\Pi_{4n-1}(U(2n))\cong\cong\mathbb{Z}$.

(ii) If n is even then any 1-section of $\tilde{p}$ is of the form $1_{\#}(\theta)-mh_{\#}(\alpha)$, where, θ, λ, α are as in part (i). Moreover, for every 1-section ψ of $U(2n)/Sp(s)$, 2ψ is of the form $1_{\#}(\varphi)$, where φ is a 2-section of $Sp(n)/Sp(s)$.

(iii) If n is odd every cross section of $U(2n)/Sp(s)$ is homotopic to a cross section of the form $1 \circ \sigma$ where σ is a cross section of $Sp(n)/Sp(s)$.

PROOF. (i) Consider the following exact sequence of the fibration $p: U(2n) \to$ $\to S^{2n-1}$,

$$\to \Pi_{4n-1}(U_{2n-1}) \to \Pi_{4n-1}(U(2n)) \xrightarrow{p_\#} \Pi_{4n-1}(S^{4n-1}) \xrightarrow{\partial_p}$$

$\Pi_{4n-2}(U(2n-1)) \to \Pi_{4n-2}(U(2n))$.

We use the following results of [1], [2] and [5].

(2.1) $\quad \Pi_{4n-1}(U(2n-1)) = 0$

(2.2) $\quad \Pi_{4n-2}(U_{2n-1}) = 0$

(2.3) $\quad \Pi_{4n-2}(U_{2n-1}) \cong \mathbb{Z}_{(2n-1)!}$

(2.4) $\quad \Pi_{4n-1}(U(2n-1)) \cong \mathbb{Z}$.

By (2.2) ∂_p is an epimorphism. Therefore the generators of $\Pi_{4n-1}(S^{4n-1})$ are mapped onto the generators of $\Pi_{4n-2}(U(2n-1) \cong \mathbb{Z}_{(2n-1)!}$ (2.3). So, the image of $p_\#$ is generated by $[(2n-1)!]\iota$. Since by (2.1) p is injective, one of the generators of $\Pi_{4n-1}(U(2n)) \cong \mathbb{Z}$ goes to $[(2n-1)!]\iota$. Call it α. Now we have

$$\tilde{p}_\#(1_\#(\theta) - mh_\#(\alpha)) = q_\#(\theta) - mp_\#(\alpha) = [(2n-1)!m+1]\iota$$

$-m[(2n-1)!]\iota = \iota$. So, $1_\#(\theta) - mh_\#(\alpha)$ is a 1-section.

(ii) First, consider the following commutative diagram

$$\begin{tikzcd}
 & & & {} & \\
 & & & \Pi_{4n-2}(Sp(n)) \arrow[u] & \\
\cdots \arrow[r] & \Pi_{4n-1}(Sp(n)/Sp(s)) \arrow[r, "1_\#"] \arrow[dr, "q_\#"'] & \Pi_{4n-1}(U(2n)/Sp(s)) \arrow[r, "t_\#"] \arrow[d, "\tilde{p}_\#"] & \Pi_{4n-1}(U(2n)/Sp(n)) \arrow[r] \arrow[u] & \cdots \\
 & \cdots & \Pi_{4n-1}(S^{4n-1}) \arrow[l] & \Pi_{4n-1}(U(2n)) \arrow[l, "p_\#"'] \arrow[u, "r_\#"'] \arrow[ul, "h_\#"'] & \cdots \arrow[l] \\
 & & & {} \arrow[u] &
\end{tikzcd}$$

where the upper row is homotopy exact sequence of the projection

$$U(2n)/Sp(s) \to U(2n)/Sp(n).$$

Since $\Pi_{4n-2}(Sp(n)) = 0$ by [2], $r_\#$ is an epimorphism.

Now assume φ is a 1-section of $U(2n)/Sp(s)$. There exists an element $u \in \Pi_{4n-1}(U(2n))$ such that

$$r_\#(u) = t_\#(\varphi).$$

Then we have

$$t_{\#}(\varphi - h_{\#}(u)) = t_{\#}(\varphi) - r_{\#}(u) = 0.$$

So, by exactness of the upper horizontal row, there exists an element $\theta \in \Pi_{4n-1}(Sp(n)/Sp(s))$ such that

$$1_{\#}(\theta) = \varphi - h_{\#}(u).$$

Since α is a generator of $\Pi_{4n-1}(U(2n)) \cong \mathbb{Z}$ (2.4), there exist $m \in Z$ such that $u = -m\alpha$. Then we have

$$\varphi = 1_{\#}(\theta) - mh_{\#}(\alpha).$$

It remains to show that θ is $(2n-1)!m+1$ section of $Sp(n)/Sp(s)$. In fact

$$q_{\#}(\theta) = \tilde{p}_{\#}(1_{\#}(\theta)) = \tilde{p}_{\#}(\varphi + mh_{\#}(\alpha)) = \tilde{p}_{\#}(\varphi) + mp_{\#}(\alpha) =$$

$$= \iota + m[(2n-1)!]\iota = [(2n-1)!m+1]\iota.$$

This completes the proof of the first statement of (ii).

Next, let ψ be a 1-section of $U(2n)/Sp(s)$. Since $\Pi_{4n-1}(U(2n)/Sp(n)) \cong \mathbb{Z}_2$ for n even ([2]), we have

$$t_{\#}(2\psi) = 0.$$

Hence there exists an element $\varphi \in \Pi_{4n-1}(Sp(n)/Sp(s))$ such that

$$1_{\#}(\varphi) = 2\psi$$

by exactness of the upper horizontal row of the diagram above. Now, φ must be a 2-section of $Sp(n)/Sp(s)$ because

$$q_{\#}(\varphi) = \tilde{p}_{\#}(1_{\#}(\varphi)) = \tilde{p}(2\psi) = 2.$$

This completes the proof of part (ii).

(iii) Since $\Pi_{4n-1}(U(2n)/Sp(n)) \cong 0$ for n odd, ([2]), $1_{\#}: \Pi_{4n-1}(Sp(n)/Sp(s)) \to \Pi_{4n-1}(U(2n)/Sp(n))$ is surjective. If β is a cross section $U(2n)/Sp(s)$ then there exists an element γ in $\Pi_{4n-1}(Sp(n)/Sp(s))$ with $1_{\#}(\gamma) = [\beta]$. Then

$$q_{\#}(\gamma) = \tilde{p}_{\#}(1_{\#}(\gamma)) = \tilde{p}_{\#}([\beta]) = \iota.$$

So, γ is a 1-section of $Sp(n)/Sp(s)$. By homotopy covering theorem γ is homotopic to a cross section σ of $Sp(n)/Sp(s)$. Therefore we have

$$\beta \simeq 1 \circ \gamma \simeq 1 \circ \sigma$$

as asserted.

REMARK. If n is even and σ is a cross section of $Sp(n)/Sp(s)$, then $1 \circ \sigma$ is a cross section but unlike the case n is odd not all cross sections are of this form. To see this it is sufficient to exhibit a 1-section φ of $U(2n)/Sp(s)$ such that $t_{\#}(\varphi) \neq 0$ in the diagram of the proof of part (ii). Since we have

$$\Pi_{4n-1}(U(2n)/Sp(n)) = \mathbb{Z}_2,\ \Pi_{4n-2}(Sp(n)) = 0,\ \Pi_{4n-1}U(2n) \cong \mathbb{Z},$$

$r_{\#}$ is onto and the generators of $\Pi_{4n-1}(U(2n))$ go to the generator of $\Pi_{4n-1}(U(2n)/Sp(n))$. Let α be the generator of $\Pi_{4n-1}(U(2n))$ such that $p_{\#}(\alpha)=[(2n-1)!]\imath$ (see the proof of part (i)). Choose any 1-section γ of $Sp(n)/Sp(s)$. Then $[(2n-1)!+1]\,\gamma$ is a $(2n-1)!+1$ section of $Sp(n)/Sp(s)$ and by (i) of Theorem 1 the element $\varphi=1_{\#}([(2n-1)!+1)]\gamma)-h_{\#}(\alpha)$ is a 1-section of $U(2n)/Sp(s)$. However,

$$t_{\#}(\psi)=-t_{\#}h_{\#}(\alpha)=-r_{\#}(\alpha)=1\neq 0.$$

3. ALMOST QUATERNION s-SUBSTRUCTURES AND QUATERNIONIC $(n-s)$-FRAMES

As defined in [6] an almost-quaternion s-substructure on the canonical $\mathbb{C}^{m-1}$-bundle ξ_{m-1} over S^{2m-1} is a $4s$-dimensional subbundle η of the underlying real bundle $r\xi_{m-1}$ of ξ_{m-1} together with normalized almost-complex $2s$-substructure $G:E(\eta)\to E(\eta)$ defined on the total space of η such that $IG=-GI$ holds. Here I is the restriction of the complex structure of $\mathbb{R}^{2m}$ induced by the multiplication by the complex number i, to $E(\eta)$.

Each almost-quaternion s-substructure whose underlying bundle (considered as a complex bundle) has trivial orthogonal complement in ξ_{m-1} corresponds to a cross section of $U(m)/Sp(s)$. Let σ be a cross section of $U(m)/Sp(s)$, $x\in S^{2m-1}$, and let $\sigma(x)$ be represented by $L\in U(m)$. Then underlying real bundle η of the almost-quaternion s-substructure has fiber at x spanned by $L(e_1),\dots,L(e_{4s})$. The structure map G on this fibre of η at x can be given by

$$G_x(L(e_\alpha))=-L(e_{3\alpha}),\ G_x(L(e_{2\alpha}))=-L(e_{4\alpha})$$

$$G_x(L(e_{3\alpha}))=L(e_\alpha),\ G_x(L(e_{4\alpha}))=L(e_{2\alpha})\ \ \alpha=1,2,\dots,s.$$

If L is another representative of $\sigma(x)$, $L'=AL$ where A is in $Sp(s)$, hence the definition of G is independent of the choice of the representative.

Now, let us restrict ourselves to the case m is even, and let $m=2n$. If in the construction above we have $L\in Sp(n)\subset U(2n)$. Then G constructed is simply the quaternionic structure $J:\mathbb{R}^{4n}\to\mathbb{R}^{4n}$ induced by the multiplication by the unit quaternion j.

Once a cross section σ of $Sp(n)/Sp(s)$ in S^{4n-1} is given we can construct the almost-quaternion s-substructure on ξ_{2n-1} corresponding to the cross section $1\circ\sigma$ of Theorem 1, part (iii) as follows.

First we recall that a cross section σ of $Sp(n)/Sp(s)$ determines a $(4n-4s)$-frame of the form $(Ku_{s+1},Ju_{s+1},Iu_{s+1},u_{s+1},\dots,Ku_n,Ju_n,Iu_n,u_n)$ (according to the conventions at the introduction), where $u_n=x$ and where the triple $I:\mathbb{R}^{4n}\to\mathbb{R}^{4n}$, $J:\mathbb{R}^{4n}\to\mathbb{R}^{4n}$, $K:\mathbb{R}^{4n}\to\mathbb{R}^{4n}$ are the standart structure maps induced by the multiplication by the unit quaternions i,j,k respectively. The orthogonal complement of this frame is preserved under I, J, K. Thus the underlying real bundle η of the almost-quaternion s-substructure we want to construct has this orthogonal complement as fibre at x and G_x is the restriction of J to the fibre of η at x.

Theorem 1 (iii) says that for n odd, this type of substructures, up to homotopy, are the only ones.

4. ORDERS OF OBSTRUCTIONS

Consider the exact homotopy sequence

$$\ldots \to \Pi_{4n-1}(U(2n)/Sp(s)) \xrightarrow{\tilde{p}_\#} \Pi_{4n-1}(S^{4n-1}) \xrightarrow{\delta} \Pi_{4n-2}(U(2n-1)/Sp(s)) \to \ldots$$

of the fibration $U(2n)/Sp(s) \to S^{4n-1}$. Since the image of $\tilde{p}_\#$ is the kernel of δ, $U(2n)/Sp(s)$ has a cross section if and only if $\delta(\iota)=0$. So, we call $\delta(\iota)$ the obstruction to cross sectioning $U(2n)/Sp(s)$ over S^{4n-1}. Let $U\{2n,s\}$ denote the order of $\delta(\iota)$ if it is finite and 0 if it is infinite. The order of obstruction to cross sectioning symplectic Stiefel manifold is defined in [4] and is the subject of current research. Following the notation of James in [4] let us denote it by $X\{n,s\}$. In this section we shall express $U\{2n,s\}$ in terms of $X\{n,s\}$ and we shall give its values explicitely in cases the values of $X\{n,s\}$ are known.

THEOREM 2. For the values of $U\{2n,s\}$, the obstructions to cross sectioning $U(2n)/Sp(s)$, we have

$$U\{2n,s\} = \begin{cases} X\{n,s\} & \text{if } n \text{ is odd} \\ \text{g.c.d. } (X\{n,s\},(2n-1)!) & \text{if } n \text{ is even} \end{cases}$$

(Here g.c.d. stands for the greatest common divisor).

PROOF. First, let n be odd. In that case we have the commutative diagram.

$$\begin{tikzcd}
 & & O \arrow[d, equal] \\
\Pi_{4n-1}(Sp(n)/Sp(s)) \arrow[r] \arrow[dr, "q_\#"'] & \Pi_{4n-1}(U(2n)/Sp(s)) \arrow[r] \arrow[d, "\tilde{p}_\#"] & \Pi_{4n-1}(U(2n)/Sp(s)) \\
 & \Pi_{4n-1}(S^{4n-1}) \cong \mathbb{Z} &
\end{tikzcd}$$

Since $1_\#$ is onto, image of $q_\#$ and image of $\tilde{p}_\#$ are equal. Since $X\{n,s\}$, $U\{2n,s\}$ are the smallest integers such that $[X\{n,s\}]\iota$ and $[U\{2n,s\}]\iota$ are in the image of $q_\#$ and $p_\#$ respectively, they must be equal.

Assume n is even. We shall use the diagram in the proof of Theorem 1 (ii). We claim that Image $\tilde{p}_\# =$ Image $q_\# +$ Image $p_\#$. First let $a \in \text{Im } q_\#$, $b \in \text{Im } p_\#$. Then there exists $\sigma \in \Pi_{4n-1}(Sp(n)/Sp(s))$, $\alpha \in \Pi_{4n-1}(U(2n))$ such that $q_\#(\sigma)=a$, $p_\#(\alpha)=b$ it follows that $\tilde{p}_\#(1_\#(\sigma)+h_\#(\alpha)) = q_\#(\sigma)+p_\#(\alpha) = a+b$, so $a+b \in \text{Im } \tilde{p}_\#$. Conversely, let $c \in \text{Im } \tilde{p}_\#$ and $\tilde{p}(\varphi)=c$. Use a diagram chasing as in the proof of part (ii) of Theorem 1. Let $\upsilon \in \Pi_{4n-1}(U(2n))$ be such that $r_\#(\upsilon)=t_\#(\varphi)$. Then $t_\#(\varphi-h_\#(\upsilon))=0$. Hence there exists $\theta \in \Pi_{4n-1}(Sp(n)/Sp(s))$ with $1_\#(\theta)=\varphi-h_\#(\upsilon)$ by the exactness of the upper horizontal row in the diagram. So, $\varphi = 1_\#(\theta)+h_\#(\upsilon)$. It follows that $c=\tilde{p}_\#(\varphi)=q_\#(\theta)+p_\#(\upsilon)$. So, $c \in \text{Im } q_\# + \text{Im } p_\#$.

Now, Im $q_\#$ is generated by $[X\{n,s\}]\iota$ and Im $p_\#$ is generated by $[(2n-1)!]\iota$ as is shown in the proof of Theorem 1 (i). It follows that

$$\text{Im } \tilde{p}_\# = \{(\lambda X\{n,s\} + \mu[(2n-1)!])\iota|,\ \lambda, \mu \in \mathbb{Z}\}.$$

Therefore Im $\tilde{p}_\#$ is generated by $d\iota$, where d is the greatest common divisor of $X\{n,s\}$ and $(2n-1)!$. This completes the proof of Theorem 2.

Now, let c_s be as in Theorem 1. We have

LEMMA 1. If $\upsilon_p(n) \geq \upsilon_p(c_{s-1})$ then

$$\upsilon_p(X\{n,s\}) = \begin{cases} \upsilon_p(c_s) - \upsilon_p(n) & \text{if } \upsilon_p(n) \leq \upsilon_p(c_s) \\ 0 & \text{if } \upsilon_p(n) > \upsilon_p(c_s) \end{cases}$$

PROOF. The proof of this lemma is the same as the proof of Corollary 5.4 in [3]. Instead of the complex Hopf line bundle over $P_{s-1}(\mathbb{C})$, the quaternionic Hopf line bundle over $P_{s-1}(\mathbb{H})$ should be used.

THEOREM 3. If $\upsilon_p(n) \geq \upsilon_p(c_{s-1})$ then

$$U\{2n,s\} = \begin{cases} \upsilon_p(c_s) - \upsilon_p(n) & \text{if } \upsilon_p(n) \leq \upsilon_p(c_s),\ n \text{ odd} \\ \max\ \{\upsilon_p(c_s) - \upsilon_p(n), \upsilon_p((2n-1)!)\} & \text{if } \upsilon_p(n) \leq \upsilon_p(c_s),\ n \text{ even} \\ 0 & \text{if } \upsilon_p(n) > \upsilon_p(c_s) \end{cases}$$

PROOF. This follows from Theorem (2) and Lemma (1).

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